Exercises

1. Let A, B, X be sets with the following properties:

   $A \subseteq X$ and $B \subseteq X$

   For any set Y if $A \subseteq Y$ and $B \subseteq Y$ then $X \subseteq Y$.

   Show that $X = A \cup B$.

2. Let $A, B \subseteq E$. Show that $A \cap B = \emptyset$ if and only if $A \subseteq B^c$. Show that $A \cup B = E$ if and only if $A^c \subseteq B$.

3. Given $A, B \subseteq E$, show that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.

4. Give examples of sets A, B, C such that $(A \cup B) \cap C \neq A \cup (B \cap C)$.

5. Show that $A = B$ if and only if $(A \cap B^c) \cup (A^c \cap B) = \emptyset$.

6. Given two sets A, B we define the *symmetric difference* $A \Delta B$ by

   $$A \Delta B = (A - B) \cup (B - A).$$

   Prove that if $A \Delta B = A \Delta C$, then $B = C$.

7. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

8. Show that a function $f : A \to B$ is injective if and only if $f(A - X) = f(A) - f(X)$ for every $X \subseteq A$.

9. Let $f : A \to B$ be given. Show that

   a. For every $Z \subseteq B$, we have $f(f^{-1}(Z)) \subseteq Z$.

   b. $f(x)$ is surjective if and only if $f(f^{-1}(Z)) = Z$ for every $Z \subseteq B$.

10. Given a family of sets $(A_\lambda)_{\lambda \in L}$, let X be a set with the following properties:

    1. For every $\lambda \in L$, we have $A_\lambda \subseteq X$.
    2. If $A_\lambda \subseteq Y$ for every $\lambda \in L$, then $X \subseteq Y$.

    Show that $X = \bigcup_{\lambda \in L} A_\lambda$.

11. Let $f : \mathcal{P}(A) \to \mathcal{P}(A)$ be a function such that if $X \subseteq Y$ then $f(Y) \subseteq f(X)$ and $f(f(X)) = X$. Show that $f(\bigcup_{\lambda \in L} X_\lambda) = \bigcap f(X_\lambda)$ and $f(\bigcap_{\lambda \in L} X_\lambda) = \bigcup f(X_\lambda)$. [Here X, Y, X_λ are subsets of A]

12. Let $\mathcal{F}(X; Y)$ denote the set of all functions with domain X and codomain Y. Given the sets A, B, C, show that there is a bijection

    $$\mathcal{F}(A \times B; C) \to \mathcal{F}(A; \mathcal{F}(B; C)).$$